Bridge position and the representativity of spatial graphs

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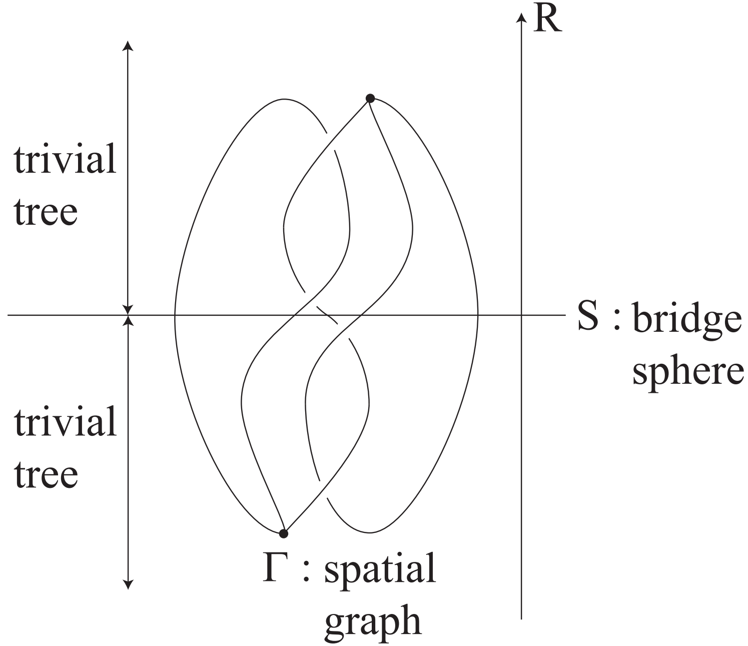
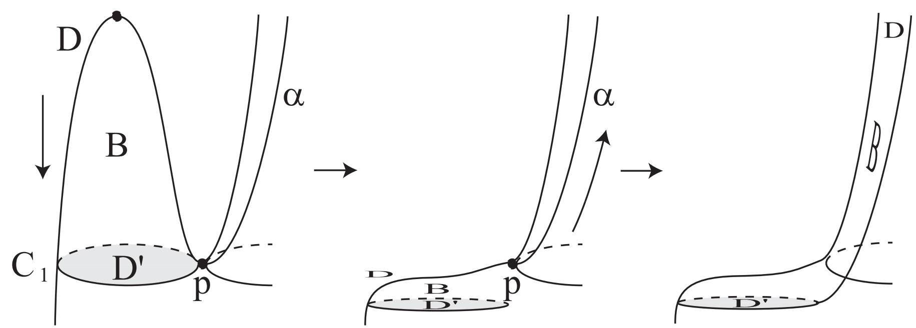
Definition (Bridge position of spatial graphs)



$F \supset \Gamma$: a closed surface

Lemma (Essential Morse position)

(F, Γ) can be isotoped so that F has no inessential saddle point.



Theorem (Otal)

Any two n-bridge positions of the trivial knot are isotopic.

Theorem 1

Let Γ be in a bridge position. Then Γ is trivial if and only if there exists a 2-sphere F containing Γ such that F intersects the bridge sphere S in a single loop.

Γ : a non-trivial spatial graph

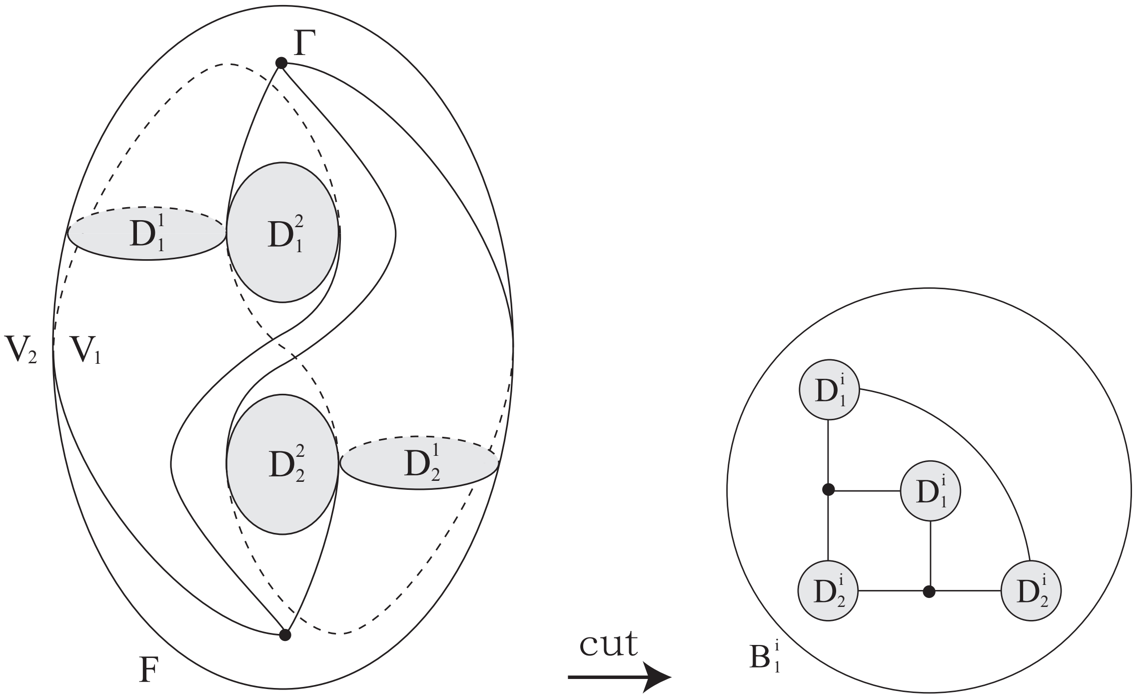
Definition

We define the *representativity* of (F, Γ) as

$$r(F, \Gamma) = \min_{D \in \mathcal{D}_F} |\partial D \cap \Gamma|$$

where $\mathcal{D}_F$ is the set of all compressing disks for F in S^3.

Example 1



$\therefore r(F, \Gamma) = 2$

Definition

We define the *representativity* of Γ as

$$r(\Gamma) = \max_{F \in \mathcal{F}} r(F, \Gamma)$$

where $\mathcal{F}$ is the set of all closed surfaces containing Γ.

Definition

We define the *bridge string number* of Γ as

$$bs(\Gamma) = \min_{\Gamma \in \mathcal{BP}_\Gamma} |\Gamma \cap S|$$

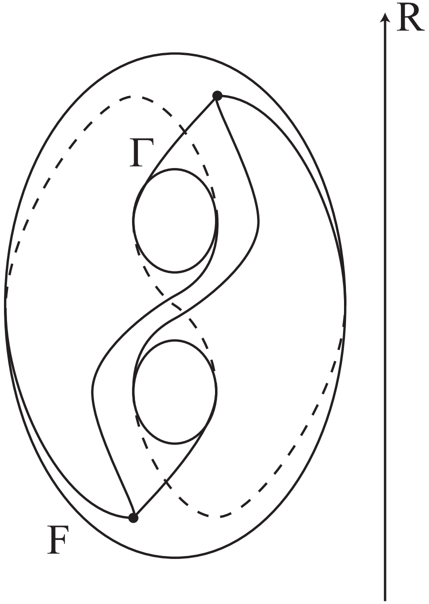
where $\mathcal{BP}_\Gamma$ is the set of all bridge position of Γ.

Theorem 2

For a non-trivial spatial graph Γ,

$$r(\Gamma) \leq \frac{bs(\Gamma)}{2}$$

Example 2



$$2 = r(F,\Gamma) \leqq r(\Gamma) \leqq \frac{bs(\Gamma)}{2} \leqq \frac{5}{2}$$

$$\therefore r(\Gamma) = 2$$

Proposition

1. $2 \leq r(K) \leq b(K)$ for a non-trivial knot K
2. $r(K) = \min\{p, q\}$ for a (p, q)-torus knot K
3. $r(K) = 2$ for a 2-bridge knot K

Theorem 3

1. $r(K) \leq 3$ for an algebraic knot K
2. $r(K) = 3$ for a (p, q, r)-pretzel knot K if and only if $(p, q, r) = \pm(-2, 3, 3)$ or $\pm(-2, 3, 5)$

Conjecture 1

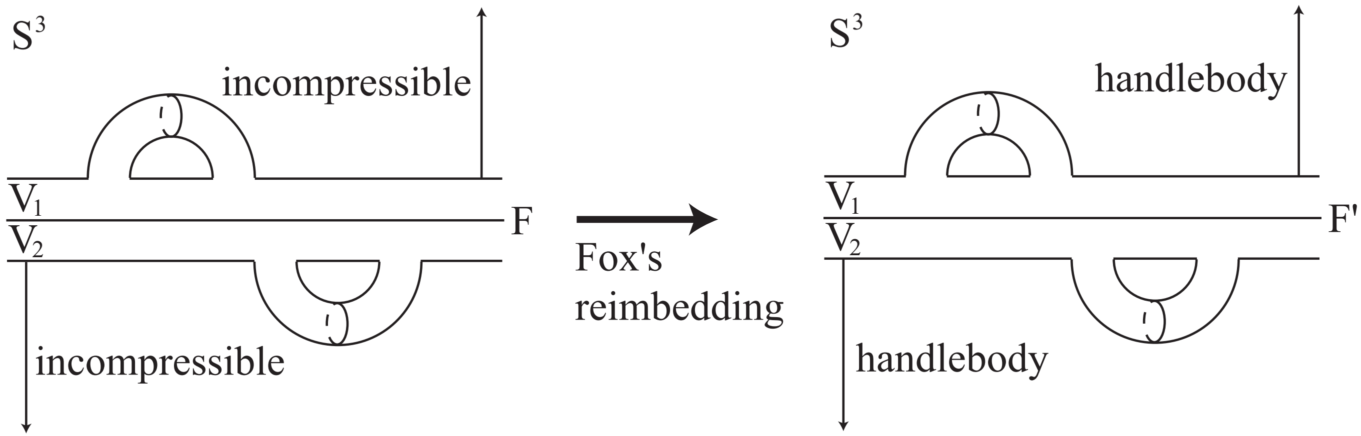
$r(K) = 2$ for an alternating knot K

Conjecture 1 is true for torus knots and Montesinos knots.

Theorem 4

For any closed surface F with $g(F) \geq g(G)$ and for any integer n, there exists a spatial graph Γ of G contained in F such that $r(F, \Gamma) \geq n$.

Proof of Theorem 4 (Idea)



V_i : the characteristic compression body for F

Definition

Γ is *totally knotted* if $\partial N(\Gamma)$ is incompressible in $S^3 - \Gamma$.

Theorem 5

If $r(\Gamma) > \beta_1(G)$, then Γ contains a connected totally knotted spatial subgraph.

Definition

Γ is *spatially n-connected* if it has no essential tangle decomposing sphere S with $|\Gamma \cap S| < n$.

Theorem 6

If $r(\Gamma) = n$, then Γ is spatially n-connected.

G : a non-planar graph

Definition

The *representativity* of (F, G) is defined as

$$r(F, G) = \min_{C \in \mathcal{C}_F} |C \cap G|,$$

where $\mathcal{C}_F$ is the set of all essential loops in F.

Definition

The *representativity* of G is defined as

$$r(G) = \max_{F \in \mathcal{F}} r(F, G),$$

where $\mathcal{F}$ is the set of all closed surfaces containing G.

Strong embedding conjecture

For a 2-connected non-planar graph G, $r(G) \geq 2$.

Strong spatial embedding conjecture

For a non-trivial spatial graph Γ of a 2-connected graph G, $r(\Gamma) \geq 2$.