Essential surfaces derived from knot and link diagrams

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November 21, 2008

1 Definitions of essential surfaces

M : orientable 3-manifold
$F \subset M$: orientable surface
$T \subset M$: 1-manifold intersecting F transversely
$\ell \subset F - T$: loop
$\alpha \subset F - T$: arc

- ℓ is *inessential* if there exists a disk $D \subset F - T$ such that $\partial D = \ell$. ℓ is *essential* if it is not inessential.
- ℓ is *meridionally inessential* if there exists a disk $D \subset F$ such that $\partial D = \ell$ and $|D \cap T| \leq 1$. ℓ is *meridionally essential* if it is not meridionally inessential.
- α is *inessential* if there exists a disk $D \subset F - T$ such that $D \cap \alpha = \partial D \cap \alpha = \alpha$ and $D \cap \partial F = \partial D - \text{int}\alpha$. α is *essential* if it is not inessential.
- F is *compressible* in $M - T$ if
  - If F is a 2-sphere, then there exists a 3-ball $B \subset M - T$ such that $\partial B = F$.
  - If F is a disk, then there exits a 3-ball $B \subset M - T$ such that $B \cap F = \partial B \cap F = F$ and $B \cap \partial M = \partial B - \text{intF}$.
  - Otherwise, there exits a disk $D \subset M - T$ such that $D \cap F = \partial D$ is essential in $F - T$.

  F is *incompressible* in $M - T$ if it is not compressible.
- F is *meridionally compressible* in (M, T) if there exists a disk $D \subset M$ such that $D \cap F = \partial D$ is meridionally essential in F and $|D \cap T| = 1$. F is *meridionally incompressible* in (M, T) if it is not meridionally compressible.
- F is *boundary-compressible* (*∂-compressible*) in M if there exists a disk $D \subset M$ such that $D \cap F = \partial D \cap F = \alpha$ is an essential arc in F and $D \cap \partial M = \partial D - \text{int}\alpha$. F is *boundary-incompressible* (*∂-incompressible*) in M if it is not boundary-compressible (∂-compressible).
- F is *boundary-parallel* (*∂-parallel*) in M if there exists an embedding $F \times I \subset M$ such that $F \times \{0\} = F$ and $(F \times I) \cap \partial M = \partial(F \times I) - \text{intF}$.

- T is *hyperbolic* in M if there exists no essential surface $S \subset M - \mathrm{intN(T)}$ with $\chi(S) \geq 0$.
- F (resp. T) is *free* in M if each component of $M - \mathrm{intN(F)}$ (resp. $M - \mathrm{intN(T)}$) is a handlebody.

Let K be a knot in the 3-sphere S^3 and $E(K)$ denotes the exterior of K. We say that an orientable surface F embedded in $E(K)$ is *essential* if it is incompressible, boundary-incompressible and not boundary-parallel. Also we say that a non-orientable surface F embedded in $E(K)$ is *essential* if the associated ∂I-bundle $F\tilde{\times}\partial I$ over F is essential.

2 σ-adequate and σ-homogeneous diagrams

Let K be a knot or link in the 3-sphere S^3 and D a connected diagram of K on the 2-sphere S^2 which separates S^3 into two 3-balls, say B_+, B_-. Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be the set of crossings of D. A map $\sigma : \mathcal{C} \to \{+, -\}$ is called a *state* for D. For each crossing $c_i \in \mathcal{C}$, we take a $+$-smoothing or $-$-smoothing according to $\sigma(c_i) = +$ or $-$. See Figure 1. Then, we have a collection of loops $l_1, \ldots, l_m$ on S^2 and call those *state loops*. Let $\mathcal{L}_\sigma = \{l_1, \ldots, l_m\}$ be the set of state loops.

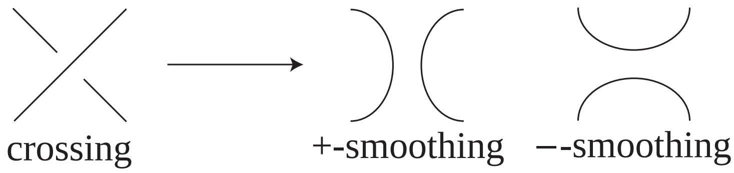


Figure 1: Two smoothings of a crossing

Each state loop l_i bounds a unique disk d_i in B_-, and we may assume that these disks are mutually disjoint. For each crossing c_j and state loops l_i, l_k whose subarcs replaced c_j by $\sigma(c_j)$-smoothing, we attach a half twisted band b_j to d_i, d_k so that it recovers c_j. See Figure 2 for $\sigma(c_j) = +$. In such a way, we obtain a spanning surface which consists of disks $d_1, \ldots, d_m$ and half twisted bands $b_1, \ldots, b_n$ and call this a *σ-state surface*

We construct a graph G_σ with signs on edges from F_σ by regarding a disk d_i as a vertex v_i and a band b_j as an edge e_j which has the same sign $\sigma(c_j)$. We call the graph G_σ a *σ-state graph.* In general, a graph is called a *block* if it is connected and has no cut vertex. It is known that any graph has a unique decomposition into maximal blocks. We say that a diagram D is *σ-adequate* if G_σ has no loop, and that D is *σ-homogeneous* if in each block of G_σ, all edges have a same sign.

3 algebraically alternating diagrams

Let K be a knot or link in the 3-sphere S^3 and $\tilde{K}$ be a diagram of K on the 2-sphere S^2. According to the Conway notation, we regard each crossing of $\tilde{K}$ as a rational tangle of slope ± 1, and sum two tangles as far as there is a bigon. After such an operation, we substitute each algebraic tangle (B, T) for a rational tangle of slope 1, -1, 0 or ∞ if the

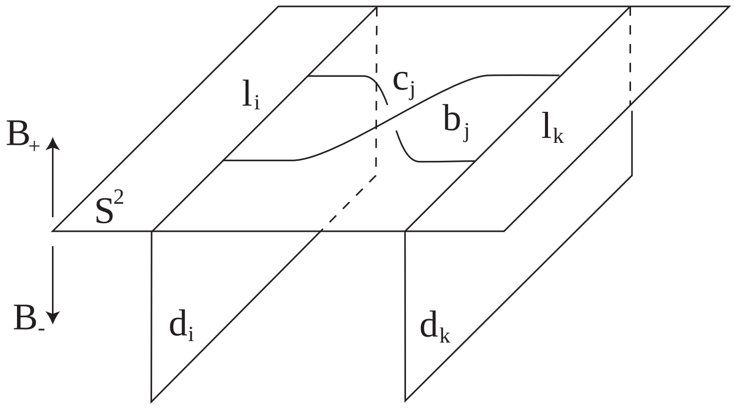
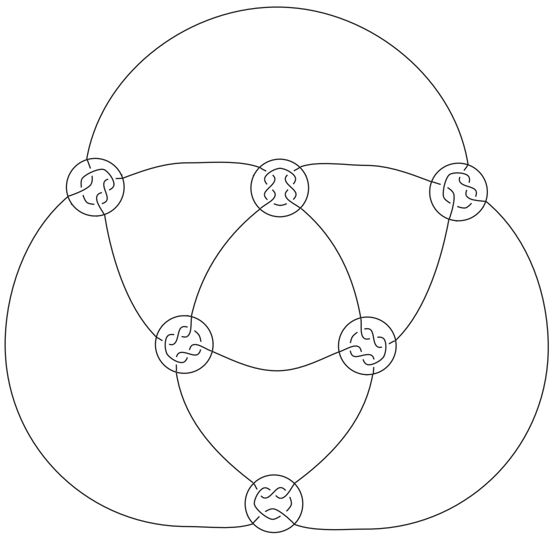
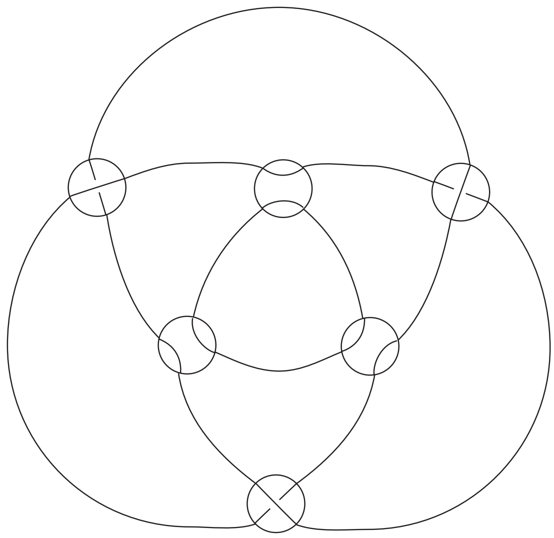


Figure 2: Recovering a crossing by a half twisted band

slope of (B, T) is positive, negative, 0 or ∞ respectively (fixing four points of ∂T). The resultant knot or link diagram is said to be *basic* and denoted by $\tilde{K}_0$. Then we say that $\tilde{K}$ is *algebraically alternating* if $\tilde{K}_0$ is alternating, and K is *algebraically alternating* if K has an algebraically alternating diagram.



$\tilde{K}$: algebraically alternating link diagram



$\tilde{K}_0$: the basic diagram of $\tilde{K}$

4 The Hasse diagram of various knot classes

5 Essential surfaces derived from knot and link diagrams

knot diagram から、本質的曲面の構成、及び曲面の位置の制限ができる。

1. 曲面の構成
   - checkerboard surface
   - Seifert surface

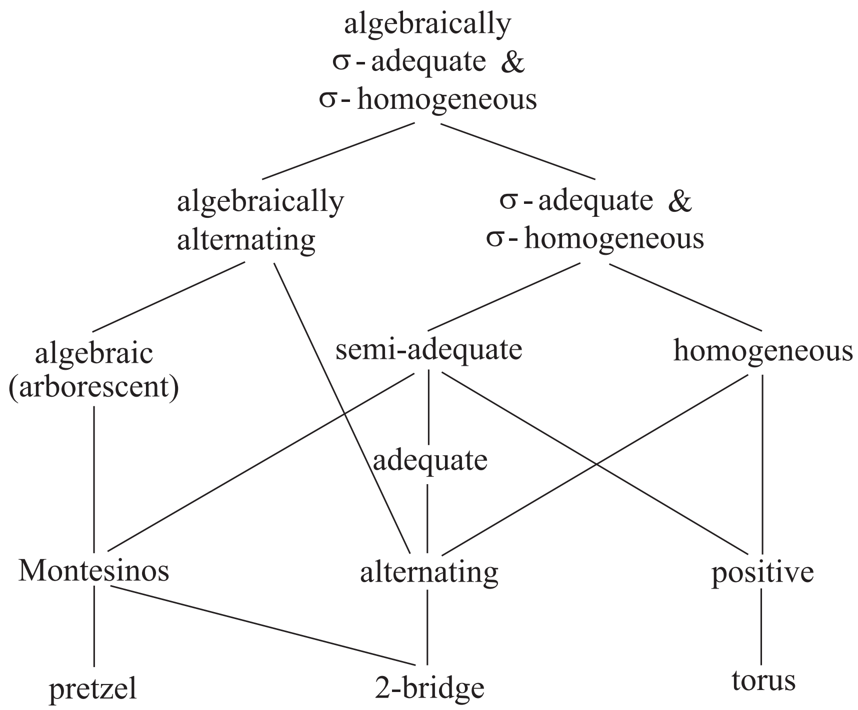


Figure 3: The Hasse diagram for the set of knot diagrams partially ordered by inclusion

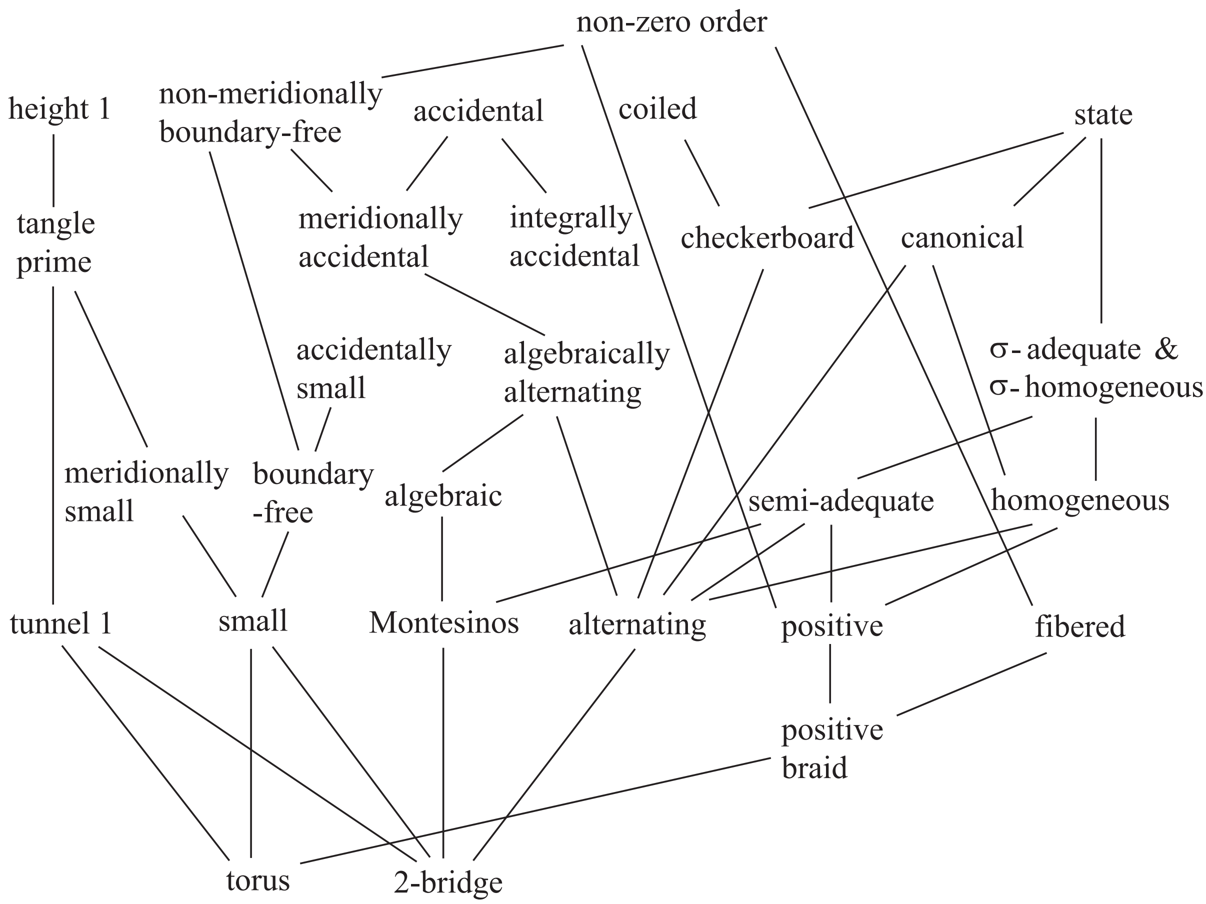


Figure 4: The Hasse diagram for the set of various knot classes partially ordered by inclusion

- state surface

2. 曲面の位置の制限

- alternating knot
- positive knot
- algebraically alternating knot

5.1 曲面の構成

Theorem 5.1 ([1]). *alternating diagram* $\Rightarrow$ *checkerboard surface* は本質的

Theorem 5.2 ([3], [6], [4]). *alternating diagram* $\Rightarrow$ *canonical Seifert surface* は本質的（最小種数）

Theorem 5.3 ([2]). *homogeneous diagram* $\Rightarrow$ *canonical Seifert surface* は本質的（最小種数）

Theorem 5.4 ([8]). σ*-adequate and* σ*-homogeneous diagram* $\Rightarrow$ σ*-state surface* は本質的

5.2 曲面の位置の制限

closed incompressible surface $F \subset S^3 - K$ の *waist* $w(F)$ を次のように定義する。

$$w(F) = \min\{\#(D \cap K) | D \text{ is a compressing disk for } F \text{ in } S^3\}$$

Theorem 5.5 ([5]). K *: alternating knot*
$F \subset S^3 - K$ *: closed incompressible surface*
$\Rightarrow w(F) = 1$

closed incompressible surface $F \subset S^3 - K$ に対して、包含写像 $i : F \to S^3 - K$ から誘導される準同型写像 $i_* : H_1(F) \to H_1(S^3 - K)$ の像 $Im(i_*)$ は、meridian で生成される巡回群 $H_1(S^3 - K)$ の部分群であるから、ある整数 m が存在して、$Im(i_*) = m\mathbb{Z}$ とおける。このとき、F の *order* を $o(F) = m$ で定義する。

Theorem 5.6 ([7]). K *: positive knot*
$F \subset S^3 - K$ *: closed incompressible surface*
$\Rightarrow o(F) \neq 0$

Theorem 5.7 ([9]). K *: algebraically alternating knot*
$F \subset S^3 - K$ *: closed incompressible surface*
$\Rightarrow w(F) = 1$

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